On Operators Preserving the Numerical Range*

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ABSTRACT

Let F be a surjective linear mapping between the algebras $L(H)$ and $L(K)$ of all bounded operators on nontrivial complex Hilbert spaces H and K respectively. For any positive integer k let $W_k(A)$ denote the kth numerical range of an operator A on H. If k is strictly less than one-half the dimension of H and $W_k(F(A)) = W_k(A)$ for all A from $L(H)$, then there is a unitary mapping $U\colon H \to K$ such that either $F(A) = UAU^*$ or $F(A) = (UAU^*)^t$ for every $A \in L(H)$, where the transposition is taken in any basis of K, fixed in advance. This generalizes the result of S. Pierce and W. Watkins on finite-dimensional spaces. The case of k greater than or equal to one-half of the dimension of H is also treated using our method. Our proofs depend on a characterization of those linear operators preserving projections of rank one, which is of independent interest.

1. INTRODUCTION

Our main result, which is stated in the abstract of this paper (cf. also Theorems 3.4, 4.2, and 4.4 below), is an extension of the result of S. Pierce and W. Watkins [10]. They solved the problem in finite-dimensional spaces, but left the question of dimension $2k$ open. The problem in infinite-dimensional spaces was partially solved by V. J. Pellegrini [9] in the case $k = 1$ under the additional assumption that F is a continuous mapping, using

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W^*-algebra techniques. In this paper we give this result in infinite dimensions without continuity assumptions on F. Moreover, we show that the same result holds when $k = 1$ and the dimension of H is two, while for $k > 1$ and the dimension of H equal to $2k$ there are two further possibilities for F, namely, we can also have either

$$F(A) = k^{-1}\operatorname{tr}(A)I - UAU^*$$

or

$$F(A) = k^{-1}\operatorname{tr}(A)I - (UAU^*)^t$$

for every $A \in L(H)$ (cf. Theorem 3.5 below).

It seems that the systematic study of this kind of problems on matrix algebras began with [6] and [7]; see also [1] and the references given there. In the last few years the interest in these problems, especially on operator algebras over infinite-dimensional spaces, has been growing again: see [2], [8], and the references given there. Our solution is based on a result on operators preserving projections of rank one. This result is given in Section 2 (see Theorem 2.1) and seems to be of some independent interest. It could be obtained from a more general Banach-space result of ours (see Theorem 2.1 of [8]), but this way of showing it would still require some tedious computations, while the complete proof can be simplified substantially and deserves to be presented here. In Section 3 the main problem is solved for degenerate operators in spaces of dimension grater than $2k - 1$; the remaining cases are treated in Section 4. Here, degenerate operator means operator of finite rank.

Throughout the paper, for any $x, y \in H$ we shall denote the scalar product of these two vectors by y^*x, while xy^* will denote the degenerate operator from $L(H)$ which sends any $z \in H$ into $x(y^*z) \in H$. For an operator $A \in L(H)$ we shall write $W_k(A)$ for the kth numerical range of A, that is, the set of all complex numbers of the form

$$\sum_{i=1}^{k} x_i^* A x_i$$

when $\{x_1, x_2, \dots, x_k\}$ runs over all possible orthonormal systems of cardinality k from H. The Hilbert-space adjoint of an operator $A \in L(H)$ is denoted by A^*; A is symmetric if $A = A^*$, and positive if $W_1(A) \subset [0, +\infty)$. A symmetric idempotent is called a projection, and a projection $P \in L(H)$ is of rank one if and only if there is a vector $x \in H$ of norm one such that $P = xx^*$.

The image and the kernel of $A \in L(H)$ are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$ respectively, while the transpose of $A \in L(H)$ in a fixed orthonormal basis of H is denoted by A^t. Note that $A = (JAJ)^*$, where J is a conjugation.

It was brought to our attention during the revision of this paper that the finite-dimensional case of our problem was solved recently by C. K. Li [5] using a different method. Moreover, the finite-dimensional case of our Theorem 2.1 could be deduced from Theorem 3 in [3] and Lemma 2 in [2]. There might be some connection between operators preserving the numerical range and those preserving the signature, although they do not appear to be related. A further study of this connection could be of some interest.

2. OPERATORS PRESERVING PROJECTIONS OF RANK ONE

Let H and K be complex Hilbert spaces of dimension strictly greater than one. In this section we study a linear mapping $F: L(H) \to L(K)$ which satisfies the following two conditions:

(A) For any two disjoint projections $P, Q \in L(H)$ of rank one, the operators $F(P), F(Q) \in L(K)$ are also disjoint projections of rank one.

(B) For any two disjoint projections $R, S \in L(K)$ of rank one, there are unique disjoint projections $P, Q \in L(H)$ of rank one such that $R = F(P)$ and $S = F(Q)$.

Here is the main result of this section (cf. [3, Theorem 3] and [2, Lemma 2]).

THEOREM 2.1. *If a linear mapping $F: L(H) \to L(K)$ satisfies conditions (A) and (B) and is continuous in the weak operator topologies on $L(H)$ and $L(K)$, then there exists a unitary mapping $U: H \to K$ such that either*

$$F(A) = UAU^*$$

or

$$F(A) = (UAU^*)^t$$

for every $A \in L(H)$, where the transposition can be taken in any basis of K, fixed in advance.

Note that in the case of transposition, the unitary operator U can, of course, depend on the choice of the basis in K which is used in transposing.

In the following lemma, a linear mapping $F: L(H) \to L(K)$ which satisfies conditions (A) and (B) is considered. A vector $x \in K$ of norm one gives rise to a projection xx^*, which yields the existence of a vector $u \in H$ of norm one such that $F(uu^*) = xx^*$. In the lemma, let u, x, and a vector $v \in H$ of norm one and orthogonal to u be fixed. Furthermore, choose a vector $y \in K$ such that $F(vv^*) = yy^*$. Observe that y is of norm one and orthogonal to x, and also that y is determined uniquely up to a multiplicative complex constant of modulus one. Denote by L_v the linear span of u and v in H, and by $U_v: L_v \to K$ the isometry defined by $U_v u = x$ and $U_v v = y$.

LEMMA 2.2. *Under the above assumptions there is a unique choice of y such that given any $A \in L(H)$ with $\operatorname{Im} A \subset L_v$ and $\operatorname{Ker} A \supset L_v^{\perp}$, we have either $F(A) = U_v A U_v^*$ or $F(A) = (U_v A U_v^*)^t$. Here, the transposition is taken in any of bases of K containing x and y.*

Proof. For any nonzero complex λ write

$$F((u+\lambda v)(u+\lambda v)^*) = y_\lambda y_\lambda^*, \tag{1}$$

where the squared norm of $y_\lambda \in K$ equals $|\lambda|^2 + 1$. If K has dimension two, y_λ is a linear combination of the vectors x and y. Otherwise, there is a nonzero $z \in K$ orthogonal to both x and y. By condition (B), there is a nonzero $w \in H$ orthogonal to both u and v and such that $F(ww^*) = zz^*$. Using condition (A), we get from the orthogonality of w and $u + \lambda v$ that also z and y_λ are orthogonal. We conclude that again y_λ is a linear combination of x and y. Therefore, we can write

$$y_\lambda = \varphi_\lambda x + \psi_\lambda y$$

for some complex numbers φ_λ and ψ_λ with $|\varphi_\lambda|^2 + |\psi_\lambda|^2 = |\lambda|^2 + 1$. From (1) we get after a short computation

$$\begin{aligned}\lambda F(vu^*) + \bar{\lambda} F(uv^*) = \left(|\varphi_\lambda|^2 - 1\right)xx^* + \varphi_\lambda\bar{\psi}_\lambda xy^* \\ + \psi_\lambda\bar{\varphi}_\lambda yx^* + \left(|\psi_\lambda|^2 - |\lambda|^2\right)yy^*.\end{aligned} \tag{2}$$

Since the operators xx^*, xy^*, yx^*, and yy^* are linearly independent, the coefficients on the right-hand side of (2) must be linear combinations of the functions λ and $\bar{\lambda}$. The first of them is bounded below and must therefore be equal to zero. The last is equal to zero as well, because it is bounded above by one. Consequently, we may choose $\varphi_\lambda = 1$, and by an appropriate (and unique) choice of y we may suppose that either $\psi_\lambda = \lambda$ or $\psi_\lambda = \bar{\lambda}$.

In the first case, $\psi_\lambda = \lambda$, we get

$$\begin{aligned} F((u+\lambda v)(u+\lambda v)^*) &= (x+\lambda y)(x+\lambda y)^* \\ &= U_v(u+\lambda v)(u+\lambda v)^* U_v^*, \end{aligned}$$

where $U_v : L_v \to K$ is defined as in the lemma. In the second case, $\psi_\lambda = \bar{\lambda}$, we get

$$\begin{aligned} F((u+\lambda v)(u+\lambda v)^*) &= (x+\bar{\lambda} y)(x+\bar{\lambda} y)^* \\ &= ((x+\lambda y)(x+\lambda y)^*)^t \\ &= (U_v(u+\lambda v)(u+\lambda v)^* U_v^*)^t, \end{aligned}$$

where the transposition is taken in a basis of K which contains x and y. It follows by linearity that the lemma holds for symmetric operators A and as a consequence for arbitrary A. ■

The following proposition is a key to our main results. Note also that Theorem 2.1 is an immediate consequence of this proposition.

PROPOSITION 2.3. *If a linear mapping $F: L(H) \to L(K)$ satisfies conditions* (A) *and* (B), *then there exists a unitary operator $U: H \to K$ such that either*

$$F(A) = UAU^*$$

or

$$F(A) = (UAU^*)^t$$

for every degenerate operator $A \in L(H)$, where the transposition can be taken in any basis of K fixed in advance.

In the proof of this proposition we shall study separately two cases. The vectors $x \in K$ and $u \in H$ are fixed as in the lemma.

Case I. *If there is a two-dimensional subspace L_v on which transposition does not occur, then there is a unitary mapping $U: H \to K$ such that $F(A) = UAU^*$ for every degenerate operator $A \in L(H)$.*

Proof. If $L_v = H$, we are done. Otherwise, choose any $z \in H$ orthogonal to u and such that L_z is different from L_v. Note that $L = L_v + L_z$ has dimension three, and therefore we can find a vector $w \in L$ of norm one, orthogonal to both u and v. Set U_v and U_w as in Lemma 2.2. Then we have $F(uv^*) = x(U_v v)^*$ and either $F(uw^*) = (U_w w)x^*$ or $F(uw^*) = x(U_w w)^*$. Assume the first possibility, and choose any nonzero complex numbers λ, μ with $|\lambda|^2 + |\mu|^2 = 1$. Then

$$F(u(\lambda v + \mu w)^*) = \bar{\lambda} x(U_v v)^* + \bar{\mu}(U_w w)x^*,$$

which is in contradiction with the fact, obtained from Lemma 2.2, that there is a vector $y \in K$ of norm one, orthogonal to x, and such that $F(u(\lambda v + \mu w)^*)$ is either of the form xy^* or yx^*. Consequently, the second possibility must hold. Now, the vector $z \in H$ can be written in the form $z = \lambda v + \mu w$ with complex λ, μ and nonzero λ, which yields

$$F(uz^*) = x(\lambda U_v v + \mu U_w w)^*.$$

Thus, the operator U defined by $Uu = x$ and $Uz = U_z z$ for every $z \in H$ extends to a unique isometry, defined on the whole of H. Note that $F(A) = UAU^*$ holds automatically for every symmetric operator $A \in L(H)$ of rank one. Next, it must be valid for every degenerate symmetric operator by linearity, and finally, $F(A) = UAU^*$ must be true for every degenerate operator $A \in L(H)$. Surjectivity of U is an immediate consequence of condition (B). ■

Case II. *If there is a two-dimensional subspace L_v on which transposition occurs, then there is a unitary operator $U: H \to K$ such that $F(A) = (UAU^*)^t$ for every degenerate operator $A \in L(H)$.*

Proof. Choose any orthonormal basis of K containing $x \in K$, and define $G(A) = F(A)^t$, where the transposition is taken in this basis. Note that G satisfies conditions (A) and (B). Use Case I of this proof for the function F to see that the transposition occurs in any of the two-dimensional subspaces of the form L_v. Hence, there is a subspace of this kind, on which the transposition does not occur for G. Use now Case I for G to get the desired result. ■

3. THE CASE OF DEGENERATE OPERATORS

Let H be a nontrivial complex Hilbert space, and k a fixed positive integer. We shall suppose throughout this section that the dimension of H is strictly greater than $2k - 1$. Let us begin with a characterization of projections of rank 1 on the space H, using only the first numerical range of operators.

PROPOSITION 3.1. *Let $A \in L(H)$ be positive. Then it is a scalar multiple of a projection of rank one if and only if the following is true: For any positive operators $B, C \in L(H)$ such that $A = B + C$, both B and C must be scalar multiples of A.*

Proof. Suppose that $A = rxx^*$, where the norm of $x \in H$ equals one and r is a nonnegative real number. The case $r = 0$ is clear; hence assume r to be strictly positive. Further, let $A = B + C$, where $B, C \in L(H)$ are both positive. For any $y \in H$ orthogonal to the vector x we have

$$0 = y^*Ay = y^*By + y^*Cy,$$

which forces $y^*By = y^*Cy = 0$, and consequently $B = ux^*$ and $C = vx^*$ for some $u, v \in H$. But the operators B and C are positive, which yields $u = sx$ and $v = tx$ for some nonnegative real numbers s and t.

On the other hand, suppose that A is not a scalar multiple of a projection of rank one. If it is a (strictly) positive multiple of a projection P of greater rank (the possibility $P = I$ is included), we can write $P = Q + R$, where Q and R are nonzero projections, and put $A = rP = rQ + rR$. If A is not a scalar multiple of a projection, then it has at least two strictly positive points in its spectrum, say $0 < r < s$. Let P be the spectral projection of the operator A relative to the interval $[0, r]$. Then $B = AP$ and $C = A(I - P)$ are nonzero positive operators with sum A. From $C = tA$ we get

$$tC = tA(I - P) = C(I - P) = C,$$

which forces $t = 1$, hence $C = A$, and finally $B = 0$, contradicting the assumption that B is nonzero. This proves that C is not a scalar multiple of A. ∎

It would be nice if this proposition had a simple generalization to the case of higher numerical ranges. Unfortunately, some difficulties arise in this case, but one direction goes analogously.

PROPOSITION 3.2. *Let $A \in L(H)$ be a scalar multiple of a projection of rank one, and $A = B + C$, where $B, C \in L(H)$ both have nonnegative kth numerical ranges. Then B and C are both scalar multiples of A.*

Proof. Write $A = rxx^*$, where the norm of $x \in H$ is one, and suppose that $A = B + C$, where $W_k(B) \cup W_k(C)$ is a subset of $[0, +\infty)$. We can conclude from this, as is well known, that B and C are both self-adjoint operators. Suppose that B is not positive, and let P be the spectral projection of B relative to the interval $(-\infty, 0]$. Choose any orthonormal system $x_1, x_2, \dots, x_j$ from $\operatorname{Im} P$ with $x_1^* B x_1 < 0$; then

$$\sum_{i=1}^{j} x_i^* B x_i < 0, \tag{3}$$

and consequently the index j must be strictly smaller than k. Therefore, $\operatorname{Im} P$ is of finite dimension and the orthonormal system above can be taken as a basis of this subspace. Note that zero is not in the point spectrum of the restriction of B to $\operatorname{Ker} P$, so it is not an isolated point of this spectrum; but, using (3), we obtain that it cannot be an accumulation point of it either. Therefore, there is a positive real number h such that $y^* B y \geqslant h y^* y$ for every $y \in \operatorname{Ker} P$. A similar consideration can be made for C and its spectral projection Q relative to the interval $(-\infty, 0]$, with one exception: namely, Q can be trivial in general, and the restriction of C to $\operatorname{Ker} Q$ can then have zero in its spectrum. In any case, the dimension of $\operatorname{Ker} P \cap \operatorname{Ker} Q$ is at least two, and there is a vector $y \in \operatorname{Ker} P \cap \operatorname{Ker} Q$ of norm one, orthogonal to vector x. From this we get a contradiction

$$0 = y^* A y = y^* B y + y^* C y \geqslant y^* B y \geqslant h > 0.$$

Thus, B and C are positive, and the proposition now follows by Proposition 3.1. ∎

In the other direction we have a weaker result.

PROPOSITION 3.3. *Let $k > 1$, and let $A \in L(H)$ have nonnegative kth numerical range. Let also the following be true: For any operators $B, C \in L(H)$ with nonnegative kth numerical ranges and such that $A = B + C$, both B and C must be scalar multiples of A. Then A is a nonnegative scalar multiple of either a projection of rank one or an operator of the form $I - kP$, where P is a projection of rank one.*

Proof. Suppose that A is not positive. As in the proof of Proposition 3.2, we see that A has only a finite number of nonpositive points in its spectrum. These points belong to the point spectrum, the sum of their multiplicities is less than k, and there is a positive constant h such that for every $y \in H$ orthogonal to all the eigenvectors corresponding to the nonpositive eigenvalues, we have $y^*Ay \geqslant hy^*y$. Choose a positive ε with $\varepsilon < (h - \lambda)/k$, where λ is the smallest eigenvalue of A. Define $B = \varepsilon(I - kP)$, where P is the projection on a one-dimensional subspace of the eigenspace of A relative to the smallest eigenvalue λ. It is clear that B has nonnegative kth numerical range, and we shall prove that the same is true for $C = A - B$. Let $x_1, x_2, \ldots, x_k$ be any orthonormal system on H. Further, let $x_{k+1} \in H$ be chosen orthonormal to this system and so that the linear span V of the vectors $x_1, x_2, \ldots, x_{k+1}$ contains $\operatorname{Im} P$. Next, choose another orthonormal basis $y_1, y_2, \ldots, y_{k+1}$ of V such that for $y_{k+1} \in V$ we have

$$y_{k+1}^* C y_{k+1} = \max v^*Cv,$$

where the maximum on the right is taken over all $v \in V$ of norm one. Since there is a $v \in V$ of norm one with $v^*Av \geqslant h$, this maximum is not smaller than $h - \varepsilon$. But any vector $u \in \operatorname{Im} P \subset V$ of norm 1 is an eigenvector of C with eigenvalue $\lambda + (k-1)\varepsilon$, which is strictly smaller than $h - \varepsilon$ by the assumption. Therefore, u is orthogonal to y_{k+1}, and we may suppose $y_1 = u$. A simple computation now gives the result

$$\begin{aligned} \sum_{j=1}^{k} x_j^* C x_j &= \sum_{j=1}^{k+1} y_j^* C y_j - x_{k+1}^* C x_{k+1} \\ &\geqslant \sum_{j=1}^{k} y_j^* A y_j - \varepsilon \sum_{j=1}^{k} y_j^*(I - kP) y_j \\ &= \sum_{j=1}^{k} y_j^* A y_j \geqslant 0. \end{aligned}$$

Now, if A is not a positive scalar multiple of $I - kP$, it can be written in the form $A = B + C$, where B and C are nonzero, with nonnegative kth numerical ranges. Furthermore, B is not a positive scalar multiple of A, and therefore also C cannot be a positive scalar multiple of A. Consequently, if the assumptions of the proposition hold and A is not a positive scalar multiple of $I - kP$, then it must be positive. In this case the proposition follows by Proposition 3.1. ∎

We are now in position to give our main result for degenerate operators. Besides the space H, fix any Hilbert space K. We shall prove two theorems at one stroke.

THEOREM 3.4. *Let either $k = 1$ or the dimension of H be strictly greater than $2k$, and let $F: L(H) \to L(K)$ be a surjective linear mapping for which $W_k(F(A)) = W_k(A)$ for every $A \in L(H)$. Then there is a unitary operator $U: H \to K$ such that either*

$$F(A) = UAU^*$$

or

$$F(A) = (UAU^*)^t$$

for every degenerate operator $A \in L(H)$, where the transposition is taken in any basis of K, fixed in advance.

THEOREM 3.5. *Let $k > 1$, let the dimension of H be equal to $2k$, and let $F: L(H) \to L(K)$ be a surjective linear mapping for which $W_k(F(A)) = W_k(A)$ for every $a \in L(H)$. Then there is a unitary operator $U: H \to K$ such that at least one of the four possibilities*

$$F(A) = UAU^*,$$

$$F(A) = (UAU^*)^t,$$

$$F(A) = k^{-1}\operatorname{tr}(A)I - UAU^*,$$

$$F(A) = k^{-1}\operatorname{tr}(A)I - (UAU^*)^t$$

holds for every $A \in L(H)$, where the transposition is taken in any basis of K, fixed in advance.

REMARK. Note that Theorem 3.5 holds also in the case $k = 1$. However, the four possibilities reduce in that case to the two possibilities of Theorem 3.4. Namely, $\operatorname{tr}(A)I - A = (WAW^*)^t$ holds for every operator A on a two-dimensional Hilbert space, as soon as the transposition is taken in a basis in which the unitary operator W has its matrix representation equal to

$$W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proof of theorems. Recall that the dimension of H was supposed to be strictly greater than $2k-1$. Note that the mapping F is necessarily injective, and choose any projection $P \in L(H)$ of rank one. Then $W_k(F(P)) = W_k(P) = [0,1]$. If $F(P) = B + C$ with $W_k(B)$ and $W_k(C)$ nonnegative, then $P = F^{-1}(B) + F^{-1}(C)$, where $W_k(F^{-1}(B))$ and $W_k(F^{-1}(C))$ are nonnegative, and by Proposition 3.2 there is a real number $r \in [0,1]$ such that $F^{-1}(B) = rP$ and $F^{-1}(C) = (1-r)P$. From the linearity of the mapping F we conclude that $B = rF(P)$ and $C = (1-r)F(P)$. If $k = 1$, we get that $F(P)$ is a positive multiple of a projection of rank one by Proposition 3.1 and because $W_k(F(P)) = [0,1]$, it must actually be a projection of rank one. When $k > 1$, we use Proposition 3.3 and the fact $W_k(F(P)) = [0,1]$ to get that either $F(P)$ is a projection of rank one, or it is of the form $k^{-1}I - Q$, where $Q \in L(K)$ is a projection of rank one.

We now proceed with the proof of the theorems in two directions:

Case I. For every projection $P \in L(H)$ of rank one, $F(P)$ is a projection of rank one.

Case II. $k > 1$, and there is a projection $P \in L(H)$ of rank one for which $F(P) = k^{-1}I - R$, where $R \in L(K)$ is a projection of rank one.

First, we show that in case I the mapping F satisfies condition (A) of Section 2. Let the projections $P, Q \in L(H)$ of rank one be disjoint. Write

$$F(P) = xx^* \quad \text{and} \quad F(Q) = \left[rx + (1-r^2)^{1/2}y\right]\left[rx + (1-r^2)^{1/2}y\right]^*,$$

where $x, y \in K$ are orthonormal and $r \in [0,1)$. Note that $W_k(P-Q) = [-1,1]$ and that $F(P) - F(Q)$ has a kernel of codimension two and two eigenvalues $-(1-r^2)^{1/2}$ and $+(1-r^2)^{1/2}$, which forces $W_k(F(P) - F(Q)) = [-(1-r^2)^{1/2}, +(1-r^2)^{1/2}]$ and consequently $r = 0$. We have thus seen that $F(P)$ and $F(Q)$ are disjoint, which gives condition (A) of Section 2.

Next, we consider the situation, when $k > 1$ and there are disjoint projections $P, Q \in L(H)$ of rank 1 such that their images are of the form $F(P) = k^{-1}I - xx^*$ and $F(Q) = [rx + (1-r^2)^{1/2}y][rx + (1-r^2)^{1/2}y]^*$, where the vectors $x, y \in K$ are orthonormal and $r \in [0,1]$. It is clear that $W_k(P+Q) = [0,2]$, but $F(P+Q)$ has an eigenvalue k^{-1} of codimension two, while $k^{-1} \pm (1-r^2)^{1/2}$ are two further eigenvalues, which implies $W_k(F(P+Q)) = [1-(1-r^2)^{1/2},\ 1+(1-r^2)^{1/2}]$ and consequently $r=0$. Thus, $k^{-1}I - F(P)$ and $F(Q)$ must be disjoint projections of rank one.

Also, if $k > 1$ and for two disjoint projections $P, Q \in L(H)$ of rank one we have $F(P) = k^{-1}I - xx^*$ and $F(Q) = k^{-1}I - [rx + (1-r^2)^{1/2}y][rx + (1-r^2)^{1/2}y]^*$, where the vectors $x, y \in K$ are orthonormal and $r \in [0,1)$, then necessarily $r=0$ and the projections $k^{-1}I - F(P)$ and $k^{-1}I - F(Q)$ are disjoint, which can be proved using similar arguments to the above.

We are now in a position to show that in case II, for every projection $P \in L(H)$ of rank one, $k^{-1}I - F(P) \in L(K)$ is a projection of rank one. Suppose, on the contrary, that there are disjoint projections $P, Q \in L(H)$ such that $k^{-1}I - F(P)$ and $F(Q)$ are disjoint projections of rank one. Write $P = uu^*$ and $Q = vv^*$, where $u, v \in H$ are orthonormal. Further, note that $P_1 = 2^{-1}(u+v)(u+v)^*$ and $Q_1 = 2^{-1}(u-v)(u-v)^*$ are again two disjoint projections of rank one and that $P_1 + Q_1 = P + Q$. From the fact that the operator $F(P_1) + F(Q_1) = F(P+Q)$ has an eigenvalue k^{-1} with eigenspace of codimension two, we conclude that there is no loss of generality in assuming $F(P_1) = zz^*$ with a vector $z \in K$ of norm one. Writing $k^{-1}I - F(P) = xx^*$ and $F(Q) = yy^*$, with some orthonormal vectors $x, y \in K$, we get

$$k^{-1}I - F(Q_1) = zz^* + xx^* - yy^*,$$

and this operator must be at the same time a projection of rank one, disjoint with zz^*. Apply it to z to get

$$0 = z + x(x^*z) - y(y^*z),$$

which yields

$$1 = z^*z = |y^*z|^2 - |x^*z|^2$$

and finally z must depend linearly on y. Hence, $F(P_1) = F(Q)$, contradicting the injectivity of F.

Assume now case I. Since there are projections of rank one in $L(K)$ which are images under F of projections of rank one from $L(H)$, case II cannot occur for the mapping F^{-1}, by the above considerations. Therefore, this mapping satisfies case I of this proof and consequently condition (A) of Section 2. This shows that F satisfies both conditions (A) and (B) of Section

2. Both possibilities of Theorem 3.4 and the first two possibilities of Theorem 3.5 now follow immediately by Proposition 2.3.

Suppose that the dimension of H is strictly greater than $2k$. Assuming case II will now lead us to a contradiction. Take pairwise disjoint projections $P_1, P_2, \dots, P_{k+1} \in L(H)$, and put $A = P_1 + P_2 + \cdots + P_{k+1}$. Then $W_k(A) = [0, k]$. Choose an orthonormal system $y_1, y_2, \dots, y_k \in K$ in the intersection of the kernels of the pairwise disjoint projections $k^{-1}I - F(P_j)$, $j = 1, 2, \dots, k+1$, of rank one, to get for the operator

$$F(A) = (k+1)k^{-1}I - \sum_{j=1}^{k+1} \left[k^{-1}I - F(P_j)\right]$$

that the real number

$$k+1 = \sum_{i=1}^{k} y_i^* F(A) y_i$$

lies in $W_k(F(A))$. This finishes the proof of Theorem 3.4.

Finally, suppose case II, when the dimension of H is $2k$. Note that the mapping $G: L(H) \to L(K)$ defined for $A \in L(H)$ by

$$G(A) = k^{-1} \operatorname{tr}(A) I - F(A)$$

is linear and satisfies condition (A) of Section 2 by the above considerations. Since $F(I) = I$, we get that G is injective and therefore bijective. Furthermore, the mapping G^{-1} can be obtained from F^{-1} in just the same way as G was obtained from F, namely $\operatorname{tr}(F(A)) = \operatorname{tr}(A)$, which implies that $G^{-1}(B) = k^{-1} \operatorname{tr}(B) I - F^{-1}(B)$ holds for all $B \in L(K)$. Since F^{-1} has the same properties as F, G^{-1} satisfies condition (A) and hence G satisfies condition (B) of Section 2. Therefore, case II yields the other two possibilities of Theorem 3.5. ■

Note that it is easy to check that both Theorems 3.4 and 3.5 hold also in the opposite direction (e.g. see [5]).

4. THE GENERAL CASE

Note that the Theorem 3.4 solves our problem in the case of finite (but not too small) dimension. It is somewhat surprising that we do not need assume any further conditions to get just the same result in infinite dimensions. The key to this fact is the following proposition. Throughout this section, H and K will be nontrivial complex Hilbert spaces and k a fixed positive integer.

PROPOSITION 4.1. *Let the dimension of H be infinite, and assume for a surjective linear mapping $F: L(H) \to L(K)$ that $W_k(F(A)) = W_k(A)$ for every $A \in L(H)$. If $Q \in L(H)$ is a projection of* rank k *and $B \in L(H)$ is a positive operator such that $BQ = QB = 0$, then $P = F(Q) \in L(K)$ is a projection of* rank k *and $A = F(B) \in L(K)$ is a positive operator with $AP = PA = 0$.*

Proof. Use Theorem 3.4 to obtain that P is a projection of rank k. It is clear that for a large enough real number M we have

$$\max\{r;\ r \in W_k(B + MQ)\} = kM,$$

but

$$\min\{r;\ r \in W_k(B)\} = 0.$$

Therefore, for any orthonormal basis $x_1, x_2, \dots, x_k$ of the subspace $\operatorname{Im} P \subset K$ we have

$$\sum_{i=1}^{k} x_i^*(A + MP)x_i \leqslant kM$$

for M large enough. On the other hand,

$$\sum_{i=1}^{k} x_i^* A x_i$$

is nonnegative and hence equal to zero. For any $y \in \operatorname{Ker} P$ of norm one, any real number $r \in (0,1)$, and any $i = 1, 2, \dots, k$, we get

$$0 \leqslant \left[(1 - r^2)^{1/2} x_i + ry\right]^* A \left[(1 - r^2)^{1/2} x_i + ry\right] + \sum_{\substack{j=1 \\ j \neq i}}^{k} x_j^* A xj$$

$$= r^2(y^*Ay - x_i^*Ax_i) + 2r(1 - r^2)^{1/2} \mathbb{R}\mathrm{e}(x_i^*Ay),$$

where $\mathbb{R}$e denotes the real part of a complex number. If x_i^*Ay is nonzero, we can multiply the vector y by a complex constant of modulus one to get x_i^*Ay strictly negative; this yields a contradiction when r is small enough. Thence, $x_i^*Ay = 0$ for every $i = 1, 2, \ldots, k$, which forces $Ay \in \operatorname{Ker} P$. Therefore, $\operatorname{Ker} P$ is invariant under A and so must be $\operatorname{Im} P$. Clearly, the restriction of A to the subspace $\operatorname{Im} P$ must have k eigenvalues, if counted according to their multiplicities, but the sum of all these eigenvalues must be zero. We shall prove in the sequel that all these eigenvalues are actually equal to zero, while the restriction of A to $\operatorname{Ker} P$ is positive. The proposition will then follow.

Now, let $R \in L(H)$ be any projection of rank one with $RQ = QR = R$, and let us determine the lower bound of $W_k(B + sR)$ for real numbers s. Write

$$r = \inf\{ y^*By;\ y \in \operatorname{Ker} Q,\ \|y\| = 1\},$$

and estimate for any orthonormal set $y_1, y_2, \ldots, y_k$ in H

$$\begin{aligned}
\sum_{i=1}^{k} y_i^*(B + sR)y_i &= \sum_{i=1}^{k} y_i^*(I - Q)B(I - Q)y_i + s \sum_{i=1}^{k} y_i^*Ry_i \\
&\geqslant r \sum_{i=1}^{k} y_i^*(I - Q)y_i + s \sum_{i=1}^{k} y_i^*Ry_i \\
&= kr - r \sum_{i=1}^{k} y_i^*(Q - R)y_i + (s - r) \sum_{i=1}^{k} y_i^*Ry_i \\
&\geqslant r + (s - r) \sum_{i=1}^{k} y_i^*Ry_i \geqslant \min(s, r).
\end{aligned}$$

If $s \leqslant r$, choose $y_1 \in \operatorname{Im} R$ and $y_2, \ldots, y_k$ from $\operatorname{Ker} R \cap \operatorname{Im} Q$; if $s > r$, choose $y_1, \ldots, y_{k-1}$ from $\operatorname{Ker} R \cap \operatorname{Im} Q$ and $y_k \in \operatorname{Ker} Q$, to get

$$\inf W_k(B + sR) = \min(s, r),$$

which implies

$$\inf W_k(A + sS) = \min(s, r), \tag{4}$$

for $S = F(R)$. Note that (4) actually holds for every projection $S \in L(K)$ of rank one with $SP = PS = S$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_k$ the eigenvalues of the restriction of A to the subspace $\operatorname{Im} P$. Choose S in such a way that the nonzero vectors of $\operatorname{Im} S$ are eigenvectors of A corresponding to a fixed eigenvalue λ_i. Further, put

$$p = \inf\{ y^*Ay;\ y \in \operatorname{Ker} P,\ \|y\| = 1\},$$

and note that p cannot be smaller than the maximum among the eigenvalues λ_j for $j = 1, 2, \ldots, k$. Further, denote $T = P - S$, choose an orthonormal set $y_1, y_2, \ldots, y_k \in K$, and estimate for any real number s

$$\begin{aligned}\sum_{j=1}^{k} y_j^*(A + sS)y_j &= \sum_{j=1}^{k} y_j^*\left[(I-P)A(I-P) + TAT + (\lambda_i + s)S\right]y_j \\ &\geqslant p\sum_{j=1}^{k} y_j^*(I-P)y_j + \sum_{j=1}^{k} y_j^*TATy_j + (\lambda_i + s)\sum_{j=1}^{k} y_j^*Sy_j \\ &= kp - \sum_{j=1}^{k} y_j^*(pT - TAT)y_j + (\lambda_i + s - p)\sum_{j=1}^{k} y_j^*Sy_j.\end{aligned}$$

It is clear that $pT - TAT$ is a positive degenerate operator such that the least upper bound of its kth numerical range equals the sum of its eigenvalues, that is,

$$(k-1)p - \sum_{\substack{j=1 \\ j \neq i}}^{k} \lambda_j = (k-1)p + \lambda_i.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{k} y_j^*(A + sS)y_j &\geqslant p - \lambda_i + (\lambda_i + s - p)\sum_{j=1}^{k} y_j^*Sy_j \\ &\geqslant \min(s, p - \lambda_i);\end{aligned}$$

but, taking $y_1 \in \operatorname{Im} S$, $y_2, \ldots, y_k \in \operatorname{Im} T$ when $s \leqslant p - \lambda_i$, and $y_1, \ldots, y_{k-1} \in$

$\operatorname{Im} T$, $y_{k+1} \in \operatorname{Ker} P$ when $s > p - \lambda_i$, we see that

$$\inf W_k(A + sS) = \min(s, p - \lambda_i).$$

Compare this with (4) to get $p - \lambda_i = r$ independently of λ_i. Hence, all the eigenvalues λ_i, $i = 1,2,\dots, k$, are equal and therefore equal to zero. Furthermore, $p = r$ is nonnegative. ∎

We can now give our main result. Let H be infinite-dimensional, and suppose that the mapping $F: L(H) \to L(K)$ is linear surjective and has the property $W_k(F(A)) = W_k(A)$ for every $A \in L(H)$. Further, let U be as in Theorem 3.4. Fix $x \in H$, put $R = xx^*$, and choose a projection $Q \in L(H)$ of rank k with $RQ = QR = R$. Note that necessarily $F(R)F(Q) = F(Q)F(R) = F(R)$. Next, choose any operator $C \in L(H)$, and write $A = (I - Q)C(I - Q)$. By Proposition 4.1 we have $F(Q)F(A) = F(A)F(Q) = 0$, which implies $F(R)F(A) = F(A)F(R) = 0$. Since $C - A$ is a degenerate operator, we conclude that

$$\begin{aligned} F(R)F(C)F(R) &= F(R)F(C-A)F(R) \\ &= F(R(C-A)R) = F(RCR). \end{aligned}$$

Suppose now that the first case of Theorem 3.4 occurs, and apply $F(B) = UBU^*$ to $B = R$ and $B = RCR = (x^*Cx)R$ to get

$$Uxx^*U^*F(C)Uxx^*U^* = (x^*Cx)Uxx^*U^*,$$

and from this (after we cancel Uxx^*U^*, which is nonzero)

$$x^*(U^*F(C)U)x = x^*Cx.$$

Since $x \in H$ was an arbitrary vector of norm one, we see that $U^*F(C)U = C$, and consequently $F(C) = UCU^*$ for every $C \in L(H)$.

If the second case of Theorem 3.4 occurs, use the mapping $G: L(H) \to L(K)$, defined by $G(A) = F(A)^t$ for every $A \in L(H)$, instead of F in the above considerations to obtain the desired results. We have thus proved the following theorem.

THEOREM 4.2. *Let the dimension of H be strictly greater than $2k$, and let $F: L(H) \to L(K)$ be a surjective linear mapping for which $W_k(F(A)) =$*

$W_k(A)$ for every $A \in L(H)$. Then there is a unitary operator $U: H \to K$ such that either

$$F(A) = UAU^*,$$

or

$$F(A) = (UAU^*)^t$$

for every $A \in L(H)$, where the transposition is taken in any basis of K, fixed in advance.

It is well known [7] that the same result holds also for dimensions strictly smaller than $2k$. For the sake of completeness we give here a proof of this fact using our methods. The key to this proof is the following proposition.

PROPOSITION 4.3. *Let the dimension n of the Hilbert space H be strictly greater than k and strictly smaller than $2k$. Then an n-tuple of symmetric operators $A_1, A_2, \ldots, A_n \in L(H)$ satisfies*

(a) *$\sum_{j=1}^{n} A_j = I$, and*

(b) *for every choice of indices $j_1, \ldots, j_r$ from the set $\{1, 2, \ldots, n\}$ we have*

$$W_k(A_{j_1} + \cdots + A_{j_r}) = [0, r]$$

for $r = 1$ and $r = n - k$,

if and only if there is an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ of H such that $A_i = x_i x_i^*$ for $i = 1, 2, \ldots, n$.

Proof. If the operators A_i are of the form $x_i x_i^*$, then (a) is true and (b) holds even for every r with $1 \leqslant r \leqslant n - k$. To see the converse, assume (a) and (b). Further, denote the eigenvalues of any A_i by

$$\lambda_1^{(i)} \leqslant \lambda_2^{(i)} \leqslant \cdots \leqslant \lambda_n^{(i)}, \tag{5}$$

every one written down according to its multiplicity. Use (b) for $r = 1$ to get

$$\lambda_1^{(i)} + \lambda_2^{(i)} + \cdots + \lambda_k^{(i)} = 0$$

and

$$\lambda_{n-k+1}^{(i)} + \lambda_{n-k+2}^{(i)} + \cdots + \lambda_n^{(i)} = 1.$$

This forces

$$\lambda_1^{(i)} + \lambda_2^{(i)} + \cdots + \lambda_{n-k}^{(i)} \leqslant 0,$$

since otherwise $\lambda_{n-k}^{(i)}$ would be strictly positive and $\lambda_{n-k+1}^{(i)}$ strictly negative, in contradiction with (5). For this reason $\operatorname{tr}(A_i) \leqslant 1$; but, using (a), we get

$$n = \operatorname{tr}(I) = \sum_{j=1}^{n} \operatorname{tr}(A_j)$$

and consequently $\operatorname{tr}(A_i) = 1$. Thence,

$$\lambda_1^{(i)} + \lambda_2^{(i)} + \cdots + \lambda_{n-k}^{(i)} = 0,$$

which requires $\lambda_{n-k}^{(i)}$ to be nonnegative. Using (5), we obtain first that $\lambda_{n-k+1}^{(i)}, \ldots, \lambda_k^{(i)}$ are all equal to zero and next that also $\lambda_1^{(i)}, \ldots, \lambda_{n-k}^{(i)}$ are zero, which implies that the operators A_i are positive.

Now, use (b) for $r = n - k$ to see that the operators $A_1, A_2, \ldots, A_{n-k}$ have a subspace L of dimension at least k in the intersection of their kernels, and write

$$P = \sum_{i=1}^{n-k} A_i.$$

Note that the dimension j of the orthonormal complement $L^{\perp}$ of L does not exceed $n - k$, and choose any orthonormal basis $x_1, x_2, \ldots, x_j$ of $L^{\perp}$ to get

$$j = \sum_{r=1}^{j} x_r^* x_r = \sum_{r=1}^{j} x_r^* P x_r + \sum_{r=1}^{j} x_r^*(I - P)x_r$$

$$= n - k + \sum_{r=1}^{j} \sum_{s=n-k+1}^{n} x_r^* A_s x_r.$$

Since $j \leqslant n - k$ and the double sum on the rightmost side above has only nonnegative terms, all of the terms must be zero and $j = n - k$ necessarily.

Therefore, the kernel of $I-P$ contains $L^{\perp}$, while the kernel of P contains L. Consequently, P is the projection on $L^{\perp}$, and the dimension of L is k.

Suppose that the intersection of kernels of $A_2,\dots,A_{n-k}$ is not strictly greater than L. From this we conclude that the kernel of the projection $A_2+\cdots+A_{n-k+1}$ equals L, and therefore $A_1=A_{n-k+1}$. Similarly we obtain that $A_1=A_j$ for $j=n-k+1,\dots,n$, and therefore $A_n=(I-P)/k$. But $A_{k+1}+\cdots+A_n=(n-k)(I-P)/k$ must again be a projection, which forces $n=2k$, contradicting the assumptions of the proposition. Thus, we can find a vector $x_1\in L^{\perp}$ of norm one which lies in the intersection of kernels of $A_2,\dots,A_{n-k}$. Because $x_1\in L^{\perp}$, this vector must lie also in the kernels of $A_{n-k+1},\dots,A_n$. For this reason

$$x_1=\sum_{j=1}^{n}A_jx_1=A_1x_1,$$

which, together with the fact that A_1 is positive and that $\operatorname{tr}(A_1)=1$, forces $A_1=x_1x_1^*$. The proposition now follows by symmetry. ■

THEOREM 4.4. *Let the dimension of H be strictly greater than k, but strictly smaller than $2k$, and let $F: L(H)\to L(K)$ be a surjective linear mapping such that $W_k(F(A))=W_k(A)$ for every $A\in L(H)$. Then there is a unitary operator $U: H\to K$ such that either*

$$F(A)=UAU^*$$

or

$$F(A)=(UAU^*)^t$$

for every $A\in L(H)$, where the transposition is taken in any basis of K, fixed in advance.

Proof. Let $P_1=x_1x_1^*$ and $P_2=x_2x_2^*$ be any disjoint projections of rank one on the space H, and choose $x_3,\dots,x_n\in H$ in such a way that the system $x_1,x_2,\dots,x_n$ is orthonormal. Then $P_i=x_ix_i^*$ for $i=1,2,\dots,n$ satisfy conditions (a) and (b) of Proposition 4.3, and so do the operators $F(P_i)$, which implies in turn that these operators are disjoint projections of rank one. Therefore, the mapping F satisfies condition (A) of Section 2. Since F^{-1} has the same properties as F, this mapping must also fulfill condition (A) of

Section 2, and for this reason F must satisfy condition (B) of Section 2. The theorem now follows from Proposition 2.3. ∎

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